CALCULUS AND GEOMETRY

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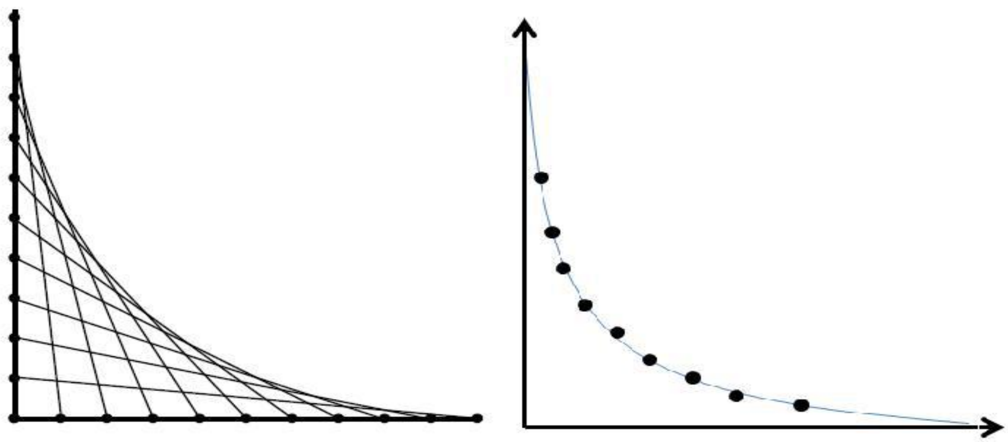
Some of the characteristic features of the ideas in calculus has to do with the concept of infinitesimals, and the corresponding infinitesimal elements, such as *dx*, *dy* and *dt*. So in order to first understand the different applications of calculus, we need to clarify what a differential means, and what it means to integrate them. A geometrical derivation of those concepts are outlined here.

First of all, let us consider a 2-dimensional plane as an example. We need to identify that there are two ways of characterizing elements on that surface, which are essentially "duals" of each other. One is, a single point is the intersection of two lines, and the other is that through two points only one line can be drawn. That means that every given curve can be defined in two ways, one which gives the points, and we can fill in the lines by "connecting the dots," and the second: as an *envelope* of lines, where the lines are given, and we "fill in" the dots.



The concept of defining a curve by its envelope is well-known, but not much applied so far, as the point-based system has been developed very well for plotting. But essentially what we are doing when we move from the envelope determined curve to the point-wise determined curve is to shift our reference... from the line at infinity to the point at zero. A similar thing can be envisioned in three dimensions, where a spherical surface, for example, is either constructed out of points on it, or out of a series of circles, while at the same time being an envelope of either planes coming from all sides, or even lines coming from all sides.

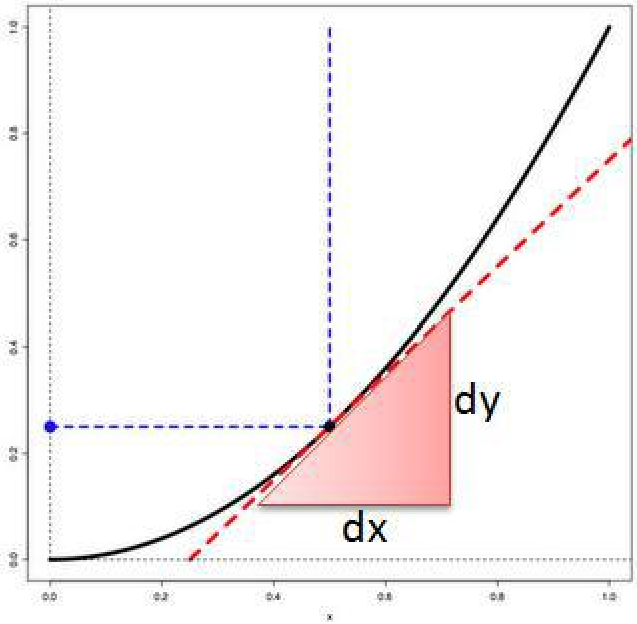
So this means we can either "build up" a surface, or even "build down" a surface. Hence there are two ways of accomplishing the same thing.

Now let us consider a curve, such as the one shown in the figure above, and identify what the traditionally known process of differentiation does with it. There is a point selected on the curve, and the differential at the point, gives us the tangent to the curve, at that point. In other words, we are getting the slope of the curve, at that point. Hence, if:

$$y = x^2;$$

$$\frac{dy}{dx} = 2x$$

where $2x$ is now its slope.



What we have converted to is an angle measure... the ratio dy/dx is actually the tangent of the angle of the slope. The geometrical transformation is from a point to a line, where we go from (x, y) to dy/dx, and from two lines (parallel to the axes) defining a point, to two “points” (dy and dx) defining a line. An inversion is accomplished.

For integration, it is the inverse process, that of building up. Hence, it is generally the transition from a line to the area under the curve, from the area to the volume in three dimensions. The logic is also the same, in that from the reference line at infinity, we are shifting back to the reference zero point, hence needing to increase the dimensionality by one in the process.

In line with that, since differentiation is a “building down” approach, it is insensitive to changes in the origin of the coordinate system. Since it is reckoned with respect to the line at infinity, every line is “equidistant” from the line at infinity, and hence, shifting around a given graph anywhere in the coordinate system will not change its differential. In other words, if the function y is replaced with $y+c$, there would be no change in the differential. This is a direct consequence of the fact that the point at zero is not fixed, unlike the unique line at infinity.

This provides the reason for the asymmetry in the behavior of the differential and the integral. Integration always results in the addition of an arbitrary constant, while differentiation always removes the effect of the arbitrary constant.

$$\int y dx = \int x^2 dx = \frac{x^3}{3} + c$$

This also provides a means of gauging whether or not the application of differential equations is valid or not in any particular system. From just the geometry, we can say: In systems where the point at zero is uniquely defined, we cannot use differential equations. Nor can we use them where the two quantities are not related with the same line at infinity. This means that in case of fluid mechanics, we are not justified in using differential equations because the geometry of the liquid aligns itself constantly with gravity, or in other words the center of the earth, and hence there is always a preferred zero point, unlike the case of solids. We also cannot use it for motion in the time region, where one component progresses while the other is fixed at unity, EXCEPT for the only function which is unchanged under the differential, i.e. the exponential. Thus it is seen that the application of quantum mechanical differential equations leads to the solutions which are combinations of exponentials, both real and imaginary.

$$y=e^x ; \frac{dy}{dx}=e^x$$

Also note that this means a reversal of the previously accepted pattern of analyzing the problem: Instead of setting up a differential equation and then trying to solve it for physical answers, we instead set up the geometry from the physics considerations, and then create the differential equation from it. This had largely been the case before calculus started being applied to mechanics during the time of Newton, when it was purely based on the kinematic aspects of velocity and acceleration. Thereafter, after Leibniz and Newton attributed a reality to differentials, it has been followed by a tendency to "find the solutions to a given differential equation." It is the exact analogue of finding the behavior of the world under a given set of forces, instead of observing that force is a property of motion. From the logic developed, it can also be predicted that the appropriate equations which should be utilized in the case of time region phenomena will have the form of *difference equations*, dealing with finite differences instead of differential equations. Liquids must be treated with rotational differentials, taken with respect to the center of the earth as the absolute frame of reference (not yet examined).

This provides the reason for the difficulty faced for the past nearly 150 years of aligning two major branches of physics into the differential equation mold… it is a question of the right geometry for the right motion, and thereby the right equation, and not the reverse.